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THE ASYMPTOTIC NUMBER OF ACYCLIC DIGRAPHS. I

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We obtain an asymptotic formula for $A_{n,q}$, the number of digraphs with n labeled vertices, q edges and no cycles. The derivation consists of two separate parts. In the first we analyze the generating function for $A_{n,q}$ so as to obtain a central limit theorem for an associated probability distribution. In the second part we show combinatorially that $A_{n,q}$ is a smooth function of q. By combining these results, we obtain the desired asymptotic formula.

1. Introduction

An acyclic digraph is a directed graph that contains no directed cycles. Let $A_{n,q}$ be the number of acyclic digraphs with n labeled vertices and q unlabeled edges. The generating function for $A_n=\sum_q A_{n,q}$ was obtained by Robinson [6] and Stanley [7]. They also obtained the asymptotic formula

$$A_n \sim \frac{n!\,2^N}{M\varrho^n}, \tag{1.1}$$

where $N=\binom{n}{2}$, $\varrho=1.48807854\ldots$ and $M=.57436237\ldots$. (See also Liskovets [4].) In fact, Robinson [6, (19)] developed a convergent series for A_n. We extend (1.1) to $A_{n,q}$ in the following theorem, but are unable to obtain a convergent series. Throughout this paper we let N denote $\binom{n}{2}$ and let $f(x,y)$ denote $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!(1+y)^N}$.

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Theorem 1. *Let $\varepsilon>0$ be given and suppose $q=q(n)$ satisfies $\varepsilon N\leqq q\leqq(1-\varepsilon)N$ for all large n. Then*

$$A_{n,q}\sim n!\binom{N}{q}\frac{e^{-w^2r}}{f(v\varrho,r)\varrho^{n+1}} \tag{1.2}$$

where

$$r=\frac{q}{N-q},\quad v=\frac{N-q}{N},\quad w=\frac{v\varrho f(v^2\varrho,r)}{2f(v\varrho,r)}$$

and $\varrho=\varrho(r)>0$ is the smallest solution of the equation $f(\varrho,r)=0$.

Properties of f which ensure the existence of $f(v\varrho,r)$, $f(v^2\varrho,r)$ and $\varrho(r)$ are deferred to the next section.

Stanley [7] observed that $A_{n,q}=M_{n,q}(-1)$, the sum of the chromatic polynomials of all the (n,q)-graphs evaluated at -1; however, (1.2) differs considerably from the asymptotics for $M_{n,q}(k)$ with $k>0$, obtained by Wright [8]. This is to be expected because the generating function for $M_{n,q}(k)$ has the form $f(x,y)^k$, which has poles when $k<0$ and not when $k>0$.

In the next section we summarize some known results concerning the generating function for $A_{n,q}$. Using these results in Section 3, we obtain a central limit theorem for the random variable $X_n(r)$ where $\Pr\{X_n(r)=m\}$ is proportional to $A_{n,m}r^m$. We do not have enough information about the generating function to obtain a local limit theorem. In Section 4 we prove combinatorially that $A_{n,q+k}\sim A_{n,q}r^{-k}$ whenever $k=o(n)$. Finally, we combine our analytic and combinatorial results to prove a local limit theorem and thus Theorem 1. Local limit theorems have been used previously to obtain asymptotic results for enumeration problems; however, this is the first instance we are aware of where it was found necessary and possible to combine an analytically derived central limit theorem with a purely combinatorial argument.

2. Properties of the generating function

Define

$$A_n(y)=\sum_{q=0}^{N}A_{n,q}y^q\quad\text{and}\quad A(x,y)=\sum_{n=0}^{\infty}\frac{A_n(y)x^n}{n!(1+y)^N}.$$

Robinson [6, (10) and Cor. 1] proved that

$$A(x,y)=\frac{1}{f(x,y)}.$$

The function $f(x,y)$ is analytic at all (x,y) such that $|1+y|>1$. For convenience set $\lambda=1/(1+y)$. It is easily seen that

$$f_x(x,y)=-f(\lambda x,y)\quad\text{and}\quad f_y(x,y)=-\lambda^2x^2f(\lambda^2x,y)/2. \tag{2.1}$$

Let t be any positive real number. By results of Pólya and Schur [5] and Laguerre [3], the zeros of $f(x,t)$ are all distinct positive reals. For details see [6, pp. 256—258]. Only the case $t=1$ was dealt with there, but the extension to positive real t is immediate.

For every complex y with $|1+y|>1$, we may let $x=\varrho(y)$ be a zero of $f(x, y)$ of least absolute value (since $f(x, y)\not\equiv 0$ its set of zeros cannot have a finite limit point). For any set R of real numbers, let R_δ be those complex numbers y with $|y-r|<\delta$ for some $r\in R$.

Lemma 1. *Let* $C(y)=1/f(\lambda\varrho(y), y)\varrho(y)$ *and let* R *be a closed subinterval of* $(0, \infty)$. *There is a* $\delta>0$ *such that whenever* $y\in R_\delta$ *and* $|x|\leqq|\varrho(y)|+\delta$

$$h(x, y) := A(x, y)-\frac{C(y)}{1-x/\varrho(y)} \tag{2.2}$$

is analytic for $x\neq\varrho(y)$ *and can be extended to be analytic for* $x=\varrho(y)$.

Proof. We can assume that $|x|$ is bounded. Since f is continuous, R is compact and (as already noted) $f(x, y)$ has distinct roots for $y\in R$, it follows that $f(x, y)$ has distinct roots whenever $y\in R_\delta$ and $\delta>0$ is sufficiently small. Since the supremum over $y\in R$ of

$$\{|x|-|\varrho(y)|: x\neq\varrho(y) \quad\text{and}\quad f(x, y)=0\}$$

is positive, the same is true for the supremum over $y\in R_\delta$. Hence, for all sufficiently small δ, we have that if $y\in R_\delta$, $|x|\leqq|\varrho(y)|+\delta$ and $f(x, y)=0$, then $x=\varrho(y)$. Thus $\varrho(y)$ is well defined. Since $|\lambda|<1$, it follows from (2.1) that $f_x(\varrho(y), y)\neq 0$ and so $\varrho(y)$ is analytic by the implicit function theorem. Hence $h(x, y)$ is analytic for $x\neq\varrho(y)$.

Since $f_x(\varrho(y), y)$ and $f_y(\varrho(y), y)$ are both non-zero, the singularity of $A(x, y)$ at $x=\varrho(y)$ is a simple pole when x or y is fixed. We have

$$\lim_{x\to\varrho(y)}(1-x/\varrho(y))A(x, y) = -1/\varrho(y)f_x(\varrho(y), y) = C(y)$$

and thus h is analytic in x for fixed $y\in R_\delta$. Implicit differentiation of $f(\varrho, y)=0$ yields

$$f_x(\varrho(y), y)\varrho'(y) = -f_y(\varrho(y), y) \tag{2.3}$$

and consequently

$$\lim_{y\to y_0}(1-y/y_0)h(\varrho(y_0), y) = 0.$$

Thus h is analytic in y for fixed x. It follows by the theorem of Hartogs that h can be made analytic at $x=\varrho(y)$ (see Hörmander [2] for example). ∎

3. The central limit theorem

Let $X_n(r)$ be a random variable with $\Pr\{X_n(r)=m\}$ proportional to $A_{n,m}r^m$. The purpose of this section is to prove:

Lemma 2. *Suppose that* $q=q(n)$ *with* $0<\varepsilon<q/N<1-\varepsilon$, $r=q/(N-q)$ *and* $\varrho(r)$ *is as in Theorem 1. There is a function* $K=K(n)=o(n)$ *such that as* $n\to\infty$

$$\frac{1}{K}\sum_{m=q}^{q+K-1}\Pr\{X_n(r)=m\}\sim\frac{e^{-v^2/2}}{\sqrt{2\pi\sigma_n(r)^2}}, \tag{3.1}$$

where

$$v = \frac{nr\varrho'(r)}{\sigma_n(r)\varrho(r)}, \quad \sigma_n(r)^2 = \frac{Nr}{(1+r)^2}$$

and the rate of convergence depends on ε.

Let

$$\mu_n(r) = \frac{Nr}{1+r} - \frac{nr\varrho'(r)}{\varrho(r)}.$$

We will show that the characteristic function of

$$Y_n(r) = \frac{X_n(r) - \mu_n(r)}{\sigma_n(r)}$$

converges pointwise to $e^{-t^2/2}$. This convergence is uniform for all $r = r(n) \in R$, an arbitrary closed subinterval of $(0, \infty)$. Using a central limit theorem for $Y_n(r)$, we then establish the lemma.

We now show that the characteristic function of $Y_n(r)$ converges to $e^{-t^2/2}$. Let $g_n(t) - \log A_n(re^{it})$. The logarithm of the characteristic function of $X_n(r)$ is

$$g_n(t) - g_n(0) = g_n'(0)t + g_n''(0)t^2/2 + O(Mt^3), \tag{3.2}$$

where M is the maximum modulus of $g_n^{(3)}(u)$ for $0 \leqq u \leqq t$ and $r \in R$. By Lemma 1, the kth derivative of $A_n(y)$ is the sum, for $j = 0$ to k, of

$$n!\binom{k}{j}\left(\left(\frac{d}{dy}\right)^{k-j}(1+y)^N\right)\left(\left(\frac{d}{dy}\right)^j (C(y)\varrho(y)^{-n}) + [x^n]\left(\frac{\partial}{\partial y}\right)^j h(x, y)\right),$$

where $[x^n]a(x, y)$ is the coefficient of x^n in $a(x, y)$. Also by Lemma 1 the radius of convergence of $(\partial/\partial y)^j h(x, y)$ is at least $\varrho(y) + \delta$ for $y \in R$. For any closed subset R' of R_δ there is a $\delta_1 > 0$ such that $\varrho(y) + \delta \geqq \varrho(y)(1+\delta_1)$ for all $y \in R'$. Hence for such y the contribution to a derivative of $A_n(y)$ from h is exponentially smaller than the other term, so we may ignore it asymptotically. We find that for $re^{it} \in R'$,

$$g_n'(t) = \frac{iNr}{r+e^{-it}} - \frac{inre^{it}\varrho'(re^{it})}{\varrho(re^{it})} + O(1)$$

$$g_n''(t) = \frac{Nre^{-it}}{(r+e^{-it})^2} + O(n)$$

$$g_n^{(3)}(t) = O(n^2).$$

By (3.2),

$$g_n(t) - g_n(0) = i\mu_n(r)t - \sigma_n(r)^2 t^2/2 + o(1),$$

when $t = o(n^{-2/3})$ uniformly for $r \in R$. The characteristic function of $Y_n(r)$ is

$$\exp\left(g_n\left(\frac{t}{\sigma_n(r)}\right) - \frac{i\mu_n(r)t}{\sigma_n(r)} - g(0)\right) \sim e^{-t^2/2}$$

provided $t=o(n^{-2/3}\sigma_n(r))$. This constraint allows $t\to\infty$ since $\sigma_n(r)$ grows like n. This proves the pointwise convergence of the characteristic function, uniformly in r.

The continuity theorem [1, p. 52, Thm 2] states, in the real case, that if $d_n(t)$ is the characteristic function of a distribution function F_n and $d_n(t)$ converges pointwise to $d(t)$ then $F_n(x)$ converges for each real x to a distribution function with characteristic function $d(t)$. Hence we have a central limit theorem for $Y_n(r)$; that is, the distribution function $F_{n,r}(x)$ of $Y_n(r)$ converges pointwise in x, uniformly in y, to the distribution function for a normal distribution of mean 0 and variance 1. Since the latter function is uniformly continuous, Theorem 1 of [1, §9] implies the convergence of the distribution function is also uniform in x. Thus

$$\sum_{m=a}^{b} \Pr\{X_n(r)=m\}-\frac{1}{\sqrt{2\pi}}\int_{t(a)}^{t(b)} e^{-x^2/2}\,dx = o(1), \tag{3.3}$$

where $t(k)=\bigl(k-\mu_n(r)\bigr)/\sigma_n(r)$. This is uniform in a, b and r. There is some function $s(n)=o(1)$ such that (3.3) is valid with $o(1)$ replaced by $o\bigl(s(n)\bigr)$. Define r by $q= =Nr/(1+r)$. Let $a=q$ and $K=\bigl[n\sqrt{s(n)}\bigr]$. It follows that

$$\frac{1}{K}\sum_{m=q}^{q+K-1} \Pr\{X_n(r)=m\} = \frac{1}{K\sqrt{2\pi}}\int_{t(q)}^{t(q+K)} e^{-x^2/2}\,dx+o\bigl(s(n)/K\bigr).$$

Since $s(n)/K=o(1/n)$, $t(q+K)^2\sim t(q)^2=O(1)$ and $t(q+K)-t(q)=K/\sigma_n(r)$, the lemma follows.

4. The combinatorial argument

Our goal in this section is to prove

Lemma 3. *Let $K(n)=o(n)$ and $\varepsilon>0$ be given. If $n, q\to\infty$ in such a way that $\varepsilon< <q/N<1-\varepsilon$,*

$$A_{n,q+k}\sim\left(\frac{N-q}{q}\right)^k A_{n,q}$$

uniformly for $|k|<K(n)$ at a rate depending on $K(n)$ and ε.

We begin with some notation. The set of all ordered partitions of $\{1, 2, \dots, n\}$ is P_n. For $\pi\in P_n$, let $|\pi|=s$ be the number of blocks of π, let $B_i=B_i(\pi)$, $(i=1, \dots, s)$ be the blocks of π, and let $b_i=|B_i|$. Let

$$g(\pi)=\sum_{i=1}^{s}\binom{b_i}{2} \quad\text{and}\quad h(\pi)=g(\pi)+\sum_{i=1}^{s-1} b_i b_{i+1}.$$

Let D be an acyclic digraph with n vertices. We associate with D the ordered partition $\pi(D)$ of the vertices of D defined inductively as follows: $D_1=D$, B_i is the set of sources (vertices with degree 0) of D_i and $D_{i+1}=D_i-B_i$. The *tower* of D, $T(D)$, is the vertices of D with all the arcs from $B_i(\pi(D))$ to $B_{i+1}(\pi(D))$ for $i\geqq 1$. Let $q(T)$ denote the number of arcs in $T=T(D)$. For simplicity, we use expressions like $g(D)$ instead of $g(\pi(D))$. Note that $q(T) \leqq h(T)-g(T)$, $g(T(D))=g(D)$ and $h(T(D))=h(D)$.

We will show first that a linear bound on $g(T)$ leads to linear bounds on $h(T)$ and $q(T)$. Next we show that most acyclic digraphs have small $g(D)$. Finally, these are combined to prove the lemma.

Suppose that T is the tower of a digraph on n vertices with $g(T) \leqq C_1 n$. By the arithmetic-geometric mean inequality for b_i^2 and b_{i+1}^2,

$$b_i b_{i+1} \leqq \binom{b_i}{2}+\frac{1}{2}b_i+\binom{b_{i+1}}{2}+\frac{1}{2}b_{i+1}.$$

Thus

$$h(T)-g(T) = \sum_{i=1}^{s-1} b_i b_{i+1} < 2\sum_{i=1}^{s}\binom{b_i}{2}+n \leqq (2C_1+1)n.$$

It follows that

$$q(T) < (2C_1+1)n \quad\text{and}\quad h(T) < (3C_1+1)n. \tag{4.1}$$

Suppose that $0<\varepsilon<m/N<1-\varepsilon$ for all $n>n_0$. We will show that there is a $C_2=C_2(\varepsilon, n_0)$ such that the fraction of acyclic (n, m)-digraphs with $g(D)>C_2 n$ is less than $(1-\varepsilon)^n$. To deal with $n \leqq n_0$, we require that $C_2>n_0^2$. The number of acyclic digraphs D with $g(D)=j$ is at most

$$|P_n|\binom{N-j}{m} < n^n\binom{N-j}{m}.$$

The number with all $b_i=1$ is

$$n!\binom{N-n+1}{m-n+1} > \left(\frac{n}{e}\right)^n\left(\frac{m-n}{N}\right)^n\binom{N}{m} > \left(\frac{nm}{3N}\right)^n\binom{N}{m}.$$

Thus, when $n>n_0$, the fraction is less than

$$\left(\frac{3N}{m}\right)^n \sum_{j>C_2 n}\left(\frac{N-j}{N}\right)^m \leqq C_3(3/\varepsilon)^n e^{-\varepsilon C_2 n}, \tag{4.2}$$

where $C_3(\varepsilon)>0$. This is less than $(1-\varepsilon)^n$ for sufficiently large C_2.

We now prove the lemma. By definition and the previous paragraph,

$$A_{n,m} = \sum_T \binom{N-h(T)}{m-q(T)} \sim \sum_{g(T)\leqq C_2 n}\binom{N-h(T)}{m-q(T)} \tag{4.3}$$

uniformly at a rate depending on ε. We will show that for $g(T) \leqq C_2 n$,

$$\binom{N-h(T)}{q+k-q(T)} \sim \left(\frac{N-q}{q}\right)^k \binom{N-h(T)}{q-q(T)}$$

uniformly at a rate depending on $K(n)$, ε and C_2. Combining this with (4.3) gives the theorem. Suppose that $|k|<Cn$ and $A, B, A-B>\delta n^2$. Then by Stirling's formula and $\log(1+u)=u+O(u^2)$

$$\frac{\binom{A}{B+k}}{\binom{A}{B}} \sim \frac{B^B}{(B+k)^{B+k}} \frac{(A-B)^{A-B}}{(A-B-k)^{A-B-k}}$$

$$= \left(\frac{A-B}{B}\right)^k \left(1+\frac{k}{B}\right)^{-B-k} \left(1-\frac{k}{A-B}\right)^{-A+B+k}$$

$$= \left(\frac{A-B}{B}\right)^k \exp\left(O(k^2/n^2)\right),$$

where the implied constants depend on C and δ. With $A=N-h(T)$ and $B=m-q(T)$, we obtain from (4.1) and $\log(1+u)=u+O(u^2)$

$$\left(\frac{A-B}{B}\right)^k = \left(\frac{N-q}{q}\right)^k \exp\left(O(k^2/n^2)\right),$$

where the implied constants depend on C_1 and ε. This completes the proof.

5. Synthesis

By Lemmas 2 and 3,

$$\frac{e^{-v^2/2}}{\sqrt{2\pi}\,\sigma_n(r)} A_n(r) \sim \frac{1}{K} \sum_{m=q}^{q+K-1} A_{n,m} r^m \sim \frac{1}{K} \sum_{m=q}^{q+K-1} A_{n,q} r^q = A_{n,q} r^q.$$

From Darboux's theorem and Lemma 1, $A_n(r) \sim n!(1+r)^N \varrho^{-n} C(r)$. This is uniform in r (recall the discussion after (3.2)). To complete the proof of Theorem 1, set $r=y$, $\varrho=\varrho(r)$, $\lambda=v$, noting that $v=-nw\sqrt{r/N} \sim -w\sqrt{2r}$ by (2.1) and (2.3), and that

$$\frac{(1+r)^N r^{-q} C(r)}{\sqrt{2\pi}\,\sigma_n(r)} = \frac{N^N}{q^q (N-q)^{N-q} \varrho f(\lambda\varrho, r)} \sqrt{\frac{(1+r)^2}{2\pi N r}} \sim \binom{N}{q} \frac{1}{\varrho f(\lambda\varrho, r)}.$$

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